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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR A NONLINEAR PARABOLI--ETC(U)
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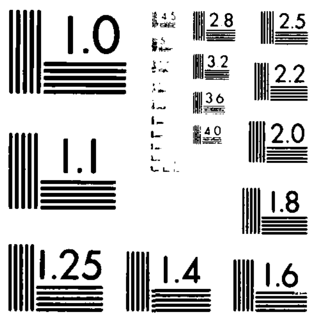

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MRC Technical Summary Report #2354

EXISTENCE OF INFINITELY MANY SOLUTIONS FOR A NONLINEAR PARABOLIC EQUATION

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March 1982

Received March 8, 1982

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

Let ϕ be a piecewise linear function which satisfies the condition $s\phi(s) \geq cs^2$, $c > 0$, $s \in \mathbb{R}$, and which is monotone decreasing on an interval $(a,b) \subset \mathbb{R}_+$. It is shown that for $f \in C^2[0,1]$ with $\max f' > a$ there exists a $T > 0$ such that the initial boundary value problem

$$u_t = \phi(u_x)_x$$
$$u_x(0,t) = u_x(1,t) = 0$$
$$u(\cdot,0) = f$$

has infinitely many solutions u satisfying

$$\|u\|_\alpha,\ \|u_x\|_\infty,\ \|u_t\|_2 \leq C(f,\phi)$$

on $[0,1] \times [0,T]$.

AMS(MOS) Subject Classifications: 35K55, 35K65

Key Words: Parabolic equation, nonlinear, diffusion, nonmonotone constitutive function, existence, nonuniqueness.

Work Unit No. 1 - Applied Analysis

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062, Mod. 1.

SIGNIFICANCE AND EXPLANATION

The equation $u_t = \phi(u_x)_x$ can be viewed as a simple model for nonlinear diffusion in one space dimension. The existing mathematical theory requires $\phi' > 0$. However, the laws of thermodynamics do not support this usual constitutive assumption for all situations. Indeed, the Clausius-Duhem inequality in one space dimension merely implies that the graph of $\phi:\mathbb{R} \to \mathbb{R}$ lies in the first and third quadrant, without necessarily requiring ϕ to be monotone nondecreasing. This raises the natural question whether the assumption $\phi' > 0$ can be replaced by the much weaker coercivity assumption and whether in this case the initial boundary value problem stated in the abstract is well-posed. This problem is apparently difficult for nonmonotone constitutive functions ϕ, since a competition between forward (whenever $\phi'(\cdot) > 0$) and backward (whenever $\phi'(\cdot) < 0$) diffusion will in general be involved, and its outcome cannot be predicted.

The purpose of this paper is to study the well-posedness of the model initial boundary value problem for the simplest case of a nonmonotone, piecewise linear, coercive ϕ which is decreasing on a single finite interval (a,b). Our result, as stated in the abstract, is that the problem has infinitely many solutions, whenever the initial function has $f' > a$, and therefore, the problem is apparently not well-posed in general. However, numerical computations suggest that there should be a natural way to single out a unique solution and it is hoped that imposing additional physical motivated assumptions will lead to a well-posed problem and further insight into the general situation of nonmonotone constitutive functions ϕ.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC and not with the author of this report.

EXISTENCE OF INFINITELY MANY SOLUTIONS FOR A NONLINEAR

PARABOLIC EQUATION

Klaus Höllig

0. INTRODUCTION. Consider the initial, boundary value problem

$$
\begin{aligned}
&u_t = \phi(u_x)_x , \quad (x,t) \in [0,1]\times[0,T] \\
(1) \qquad &u_x(0,t) = u_x(1,t) = 0 , \quad t \in [0,T] , \\
&u(x,0) = f(x) , \quad x \in [0,1] .
\end{aligned}
$$

If ϕ is strictly monotone increasing with $\phi' > c > 0$ (1) has a unique solution which is, roughly speaking, as smooth as the function ϕ. On the other hand, if $\phi' < -c < 0$, (1) is a 'backward' parabolic equation and, because of the smoothing effect, may have a solution only for very special initial values.

In nonlinear diffusion, for which equation (1) is a simple model in one space dimension, ϕ needs not to be monotone increasing. The Clausius - Duhem inequality [D, p.79] in one space dimension merely implies that the graph of ϕ lies in the first and third quadrant. An additional, physically reasonable hypothesis regarding ϕ is the coercivity condition

$$(c) \qquad s\,\phi(s) > c\,s^2 , \quad c > 0 .$$

This assumption allows ϕ to have monotone decreasing parts (e.g. the model cubic $\phi(s) = \frac{1}{3} s^3 - \frac{3}{2} s^2 + 2 s$).

A natural and interesting question is whether problem (1) has a solution, if the usual assumption $\phi' > 0$ is replaced by the weaker coercivity condition (c). Under this hypothesis J. Bona, J. Nohel and L. Wahlbin [BNW] obtained several a priori estimates for the problem (1). Assuming the existence of a solution u in $W_2^1([0,1]\times[0,T])$ they showed e.g. that

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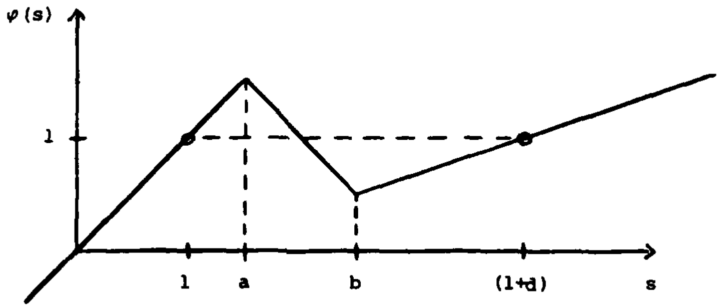
$$\|u(\cdot,T)\|^2_{2,[0,1]} + 2c\|u_x\|^2_{2,[0,1]\times[0,T]} \le \|f\|^2_{2,[0,1]}$$

$$\|u_x\|_{\infty,[0,1]\times[0,T]} \le c^{-1}\|\phi(f')\|_{\infty,[0,1]} .$$

For smooth solutions they also proved a maximum principle for u_x. Using Galerkin approximations these estimates almost yield the existence of weak solutions. The difficulty, which prevents completing an existence proof, is that if ϕ is not monotone, the map $u_x \to \phi(u_x)$ is not weakly continuous.

The above estimates and J. Nohel's continuing optimism concerning the existence of weak solutions motivated our investigations. We study the simplest case where ϕ is a typical nonmonotone piecewise linear function satisfying (c). We shall assume that ϕ is of the form

(2)



i.e. $\phi(s) = m_1 s - (m_1+m_2)(s-a)_+ + (m_2+m_3)(s-b)_+$, $m_j > 0$, $\phi(b) > 0$.

Clearly, if the initial function f satisfies $\max f' < a$, then by the maximum principle any solution of the equation $u_t = m_1 u_{xx}$ solves the problem (1).

If, however, $\max f' > a$, (1) cannot have a smooth solution in general. More precisely, <u>if</u> $f'([\alpha,\beta]) \subseteq (a,b)$ <u>for some interval</u> $[\alpha,\beta] \subseteq (0,1)$ <u>and</u> $f|_{(\alpha,\beta)}$ <u>is not analytic, then (1) cannot have a solution with continuous partial derivative with respect to</u> x.

To see this, assume that u_x is continuous on $[\alpha,\beta]\times[0,T]$ and define $v(x,t) := u(x,T-t)$. Since $u_x(\cdot,0) = f'$ we have for small enough T $u_x([\alpha,\beta],[0,T]) \subseteq (a,b)$. Hence v, restricted to the rectangle $[\alpha,\beta]\times[0,T]$ is a solution of

$$v_t = m_2 v_{xx}$$
$$v(\alpha,t) = u(\alpha,T-t) \ , \quad v(\beta,t) = u(\beta,T-t)$$
$$v(x,0) = u(x,T)$$

and the smoothing property of the heat equation implies that $v(\cdot,T) = u(\cdot,0) = f$ is analytic on (α,β) which is contrary to the hypothesis.

In fact the above argument shows that in general u_x <u>cannot be piecewise continuous with respect to a finite partition of</u> $[0,1]\times[0,T]$. This fact is also supported by numerical computations which we did jointly with C. de Boor. Approximations $u_x^n(\cdot,t)$ to $u_x(\cdot,t)$ oscillate in intervals where ϕ' is negative. This phenomenon has been independently observed by G. Strang and M. Abdel-Naby [AS].

We use the following <u>notation</u>

$$\|\psi\|_\infty = \sup_{\xi\in\Omega} |\psi(\xi)| \ , \quad \|\psi\|_p = \left(\int_\Omega |\psi(\xi)|^p \, d\xi \right)^{1/p}$$
$$\|\psi\|_\alpha = \|\psi\|_\infty + \sup_{\xi,\xi'\in\Omega} |\psi(\xi)-\psi(\xi')|/|\xi-\xi'|^\alpha$$

for the norms of the spaces C, L_p, C^α, $\alpha < 1$. Unless explicitly specified the domain Ω will be clear from the context, e.g. in the Theorem below the norms are taken on $\Omega = [0,1]\times[0,T]$.

THEOREM. <u>For</u> ϕ <u>as described by (2) and any</u> $f \in C^2[0,1]$ <u>with</u> $\max f' > a$, $f'(0) = f'(1) = 0$, <u>there exists</u> $T > 0$ <u>such that the problem (1) has infinitely many solutions</u> u <u>satisfying the equation</u> $u_t = \phi(u_x)_x$ <u>on</u> $[0,1]\times[0,T]$ <u>in the sense of</u> L_2 . <u>Each such solution satisfies the estimates</u>

$$\|u\|_\alpha \ , \quad \|u_x\|_\infty \ , \quad \|u_t\|_2 \ < \ C$$

<u>where</u> T, C, α <u>depend on</u> ϕ <u>and</u> f . <u>Moreover, we have</u>

$$u_x([0,1],(0,T]) \cap (a,b) = \emptyset \ .$$

The last conclusion reflects the qualitative behaviour of numerical solutions to the problem (1) which has been observed in numerical experiments. It also shows that <u>the solutions do not depend on the values of</u> ϕ <u>in the interval</u> (a,b) . In fact $\phi|_{(a,b)}$

could be defined arbitrarily. Also note, that the solutions are slightly smoother than predicted by the a priori estimates mentioned earlier.

Before beginning with the proof of the Theorem let us choose a convenient normalization for the piecewise linear functions ϕ. The change of variables $u(x,t) = U(px,qt)$ transforms (1) into the equation

$$U_t = \Phi(U_x)_x$$

with $\Phi(s) = q^{-1}p\,\phi(ps)$. From this one can easily check that we may without loss of generality assume that $m_1 = 1$ and $\phi(a) + \phi(b) = 2$. If we define d by $\phi(1+d) = \phi(1) = 1$, then with this normalization ϕ is completely determined by the three parameters a, b, d (cf. figure (2) where this normalization has already been chosen).

The key of the existence proof is the relation

$$\phi(s+A+Bs) = \phi(s)\ , \quad 2-a \le s \le a \tag{3}$$

where, to be precise, $A = (ad+a+b-2-2d)/(a-1)$, $B = (2+d-a-b)/(a-1) > -1$, $A+B = d > 0$. The following pathalogical feature of the problem (1) is implied by the identity (3). <u>For any function</u> $\chi : [0,1] \to \{0,1\}$ <u>and</u> $c \in [2-a,a]$

$$u(x,t) = \int_0^x \big(c + \chi(y)(A+Bc)\big)\,dy$$

<u>is a solution of</u> $0 = \phi(u_x)_x$, because by (3) $\phi(u_x) = c$.

In proving the Theorem we first consider in section 1 the special case $B = 0$ which simplifies the analysis and illustrates the basic idea behind the proof of the general case done in section 2. In an appendix we state a regularity result for a linear parabolic equation needed for our arguments.

1. THE CASE $B = 0$. For $B = 0$ the normalized ϕ is of the form

$$(2') \qquad \phi(s) = \begin{cases} s, & s \le a \\ s-d, & s \ge b \end{cases}$$

and the relation (3) becomes particularly simple

$$(3') \qquad \phi(s+d) = \phi(s), \quad 2-a \le s \le a .$$

Numerical computations indicate the existence of solutions u with $u_x([0,1],(0,T]) \cap (a,b) = \emptyset$. This suggests to split u into a smooth and an oscillating part, $u = v + w$. In view of (3') we choose w of the form

$$(4) \qquad w(x,t) = d \int_0^x \chi(y,t)\, dy$$

with $\chi : [0,1]^2 \to \{0,1\}$, i.e.

$$(4.1) \qquad w_x \in \{0,d\} .$$

If $v_x \ge 1$, $\phi(v_x+w_x)_x = v_{xx}$, i.e. the oscillations of w_x are not recognized by the right hand side of the equation (1). This is the reason for the existence of solutions corresponding to initial data f with $f'([0,1]) \cap (a,b) \ne \emptyset$. χ, and hence w, will depend only on f and constructed so that the resulting equation for v is as regular as possible. To this end, and for reasons that will become apparent in the proof of Proposition 1, we require that w satisfies

$$(4.2) \qquad w_t \in L_\infty$$

$$(4.3) \qquad w(x,0) = h(x), \text{ where}$$

$$h(0) = 0 \text{ and } h'(x) := \begin{cases} 0, & f'(x) \le 1 \\ f'(x)-1, & 1 \le f'(x) \le 1+d \\ d, & 1+d \le f'(x) \end{cases}$$

(4.4) $\qquad$ <u>for all</u> $\varepsilon > 0$ <u>there is</u> $T > 0$ <u>such that for</u> $t \le T$

$$\{f' \le 1-\varepsilon\} \subseteq \{\chi(\cdot,t) = 0\}$$

$$\{f' \ge 1+d+\varepsilon\} \subseteq \{\chi(\cdot,t) = 1\} .$$

Here we used the notation $\{\psi > y\} := \{x : \psi(x) > y\}$. Condition (4.4) means that the oscillations of χ are essentially restricted to a neighborhood of the set $\{x : f'(x) \in (1,1+d)\} \times [0,T]$.

Assuming the existence of a function w satisfying (4.1) - (4.4) we now construct a solution for the problem (1).

PROPOSITION 1. Let w of the form (4) satisfy (4.1) - (4.4) and define v as the solution of the problem

$$
(5)\qquad
\begin{aligned}
&v_t = v_{xx} - w_t\\
&v_x(0,t) = v_x(1,t) = 0\\
&v(x,0) = g(x)\ ,\ \text{where}\\
&g(0) = f(0) \quad \text{and} \quad g'(x) := \begin{cases} f'(x) , & f'(x) \leq 1 \\ 1 , & 1 \leq f'(x) \leq 1+d \\ f'(x)-d , & 1+d \leq f'(x) \end{cases} .
\end{aligned}
$$

Then there exists $T > 0$ such that $u = v + w$ is a solution of (1) satisfying the regularity assertions of the Theorem.

Proof. From the definition of the initial values g, h for v, w and (4.4), (5) it follows that u satisfies the boundary and initial conditions. Also note that $g'', h'' \in L_\infty$. This follows from the continuity of g', h' and $f \in C^2[0,1]$.

In view of (5) the equation $u_t = \phi(u_x)_x$ is equivalent to

$$
(*)\qquad \phi(v_x+w_x)_x = v_{xx} .
$$

Since $g'', w_t \in L_\infty$ we have by Theorem A (cf. appendix), applied to the problem (5), that $v_x \in C^\alpha$. Therefore, for small t,

$$
\{v_x(\cdot,t) > 1+\varepsilon\} \subseteq \{g' > 1+\varepsilon/2\} = \{f' > 1+d+\varepsilon/2\} .
$$

From (2') and (4.4) it follows that for $x \in \{v_x(\cdot,t) > 1+\varepsilon\}$

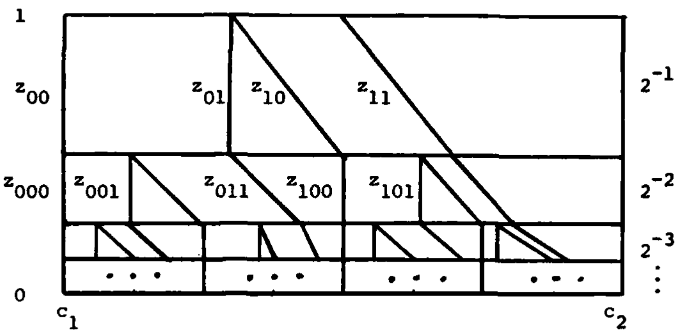
$$
\phi(v_x(x,t) + w_x(x,t)) = \phi(v_x(x,t) + d) = v_x(x,t) .
$$

Similarly we argue if $x \in \{v_x(\cdot,t) < 1-\varepsilon\}$. Finally, for $x \in \{|v_x(\cdot,t) - 1| < 2\varepsilon\}$ with $\varepsilon < (a-1)/2$ we apply (3') and (4.1) to complete the proof of (*).

The regularity assertions for u stated in the Theorem are consequences of (4.1), (4.2) and Theorem A. |||

It remains to construct a function χ so that $w = d \int \chi$ satisfies (4.1) - (4.4). Let $[c_1,c_2] \subseteq [0,1]$ be any interval containing supp h' and consider the following infinite partition $\Pi(c_1,c_2,h)$ of $[c_1,c_2]\times[0,1]$

(6)



where we denote by z_r , $r = r_1 r_2 \cdots r_{|r|}$, $r_\nu \in \{0,1\}$, the lines

$$t \to x = z_r(t) \; : \; [2^{-|r|+1}, 2^{-|r|+2}] \to [0,1] \; .$$

The endpoints of these lines are denoted by $(\underline{z}_r, 2^{-|r|+1})$, $(\bar{z}_r, 2^{-|r|+2})$ respectively. We write rs, $s \in \{0,1\}$, for $r_1 r_2 \cdots r_{|r|} s$. Whenever it is convenient we interpret r as the dual number $\Sigma_{\nu=1}^{|r|} r_\nu \, 2^{\nu-1}$, e.g. we write $r1 = r0+1$ etc. However, since we do not ignore leading zeros in the sequence r , different r's may correspond to the same number. As indicated by figure (6) we have

(7.1) $\qquad \cdots < z_r < z_{r+1} < \cdots$

(7.2) $\qquad \underline{z}_{r0} = \bar{z}_{r00} = z_{r00} = c_1 + r\, 2^{-|r|} (c_2 - c_1)$

(7.3) $\qquad \underline{z}_{r1} = \bar{z}_{r11}$

(7.4) $\qquad \bar{z}_{r01} = \bar{z}_{r10} \in [\underline{z}_{r0}, \underline{z}_{r1}]$.

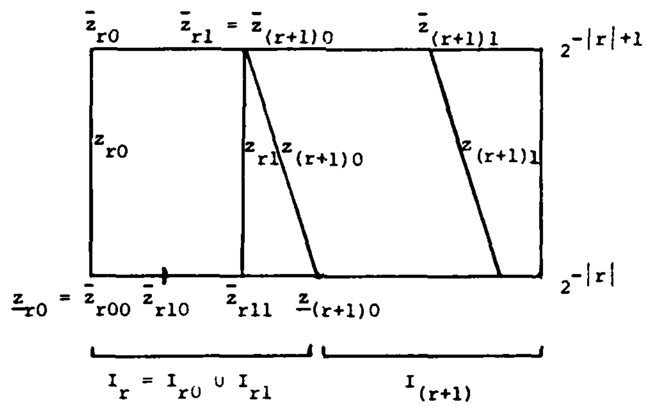
With this notation we define χ by

(8)
$$\chi(x,t) = \begin{cases} 1 \, , & z_{r0}(t) < x < z_{r1}(t) \, , \quad 2^{-|r|} < t < 2^{-|r|+1} \\ 0 \, , & \text{otherwise} \end{cases} \; .$$

We choose now the points $\bar{z}_{r1}$ which determine the partition Π so that w interpolates h on the vertical lines z_{r00}, more precisely

$$w(\underline{z}_{r0},t) = h(\underline{z}_{r0}) \ , \quad t \leq 2^{-|r|} \ . \tag{9}$$

Clearly, this implies $\lim_{t \to 0} w(\cdot,t) = h$, i.e. (4.3). Consider a typical subrectangle of the partition Π



We define $\bar{z}_{r1}$ by

$$d\,(\bar{z}_{r1} - \bar{z}_{r0}) = \int_{I_r} h'(y)\,dy \ , \quad I_r := [\underline{z}_{r0}, \underline{z}_{(r+1)0}] \tag{10}$$

and note that $\bar{z}_{r0}$ is implicitly determined by (7.2) and (7.4).

Since $0 \leq h' \leq d$ and $I_{r0} \cup I_{r1} = I_r$ we have

$$\bar{z}_{r11} - \bar{z}_{r00} = d^{-1} \left(\int_{I_{r0}} h' + \int_{I_{r1}} h' \right) \leq \underline{z}_{(r+1)0} - \underline{z}_{r0} \ ,$$

e. $\underline{z}_{r1} = \bar{z}_{r11} \leq \bar{z}_{(r+1)00} = \underline{z}_{(r+1)0}$ which is consistent with (7.1). Moreover, we see that

$$d\,(\underline{z}_{r1} - \underline{z}_{r0}) = d\,(\bar{z}_{r11} - \bar{z}_{r00}) = \int_{I_r} h' = d\,(\bar{z}_{r1} - \bar{z}_{r0}) \ ,$$

i.e. the lines z_{r0}, z_{r1} are parallel. Therefore we have for $t \in [2^{-|r|-1}, 2^{-|r|}]$

$$\int_{I_r} w_x(y,t)\,dy = d\,(\bar{z}_{r11} - \bar{z}_{r00}) = \int_{I_r} h'$$

which implies (9). Note that we have equality in (7.1) in either one of the following cases

$$
\begin{aligned}
& z_{r0} = z_{r1} \text{, iff } h'(I_r) = 0 \\
(11) \qquad & z_{r01} = z_{r10} \text{, iff } h'(I_{r0}) = d \\
& z_{r11} = z_{(r+1)00} \text{, iff } h'(I_r) = d .
\end{aligned}
$$

We already saw that w, defined by (4) and implicitly by (10), satisfies (4.1) and (4.3). From (4) and figure (6) it is clear that w is continuous and therefore it is sufficient to compute w_t on the rectangles $[0,1]\times(2^{-j},2^{-j+1})$. For $2^{-j} < t < 2^{-j+1}$ we have

$$(12) \qquad w(x,t) = \sum_{|r|=j} d\left((x - z_{r0}(t))_+ - (x - z_{r1}(t))_+ \right)$$

$$(13) \qquad w_t(x,t) = \sum_{|r|=j} d\left(z'_{r1}\,(x - z_{r1}(t))^0_+ - z'_{r0}\,(x - z_{r0}(t))^0_+ \right) .$$

By (7.1), $\bar{z}_{r00} = z_{r00} \leq z_{r01} \leq z_{r10} \leq z_{r11} \leq \bar{z}_{(r+1)00}$, with $\bar{z}_{(r+1)00} - \bar{z}_{r00} = 2^{-|r|}(c_2-c_1)$, which implies

$$(7.5) \qquad 0 \leq z'_{r0} = z'_{r1} \leq 2 .$$

This, together with (13), shows $\|w_t\|_\infty \leq 2$.

To prove (4.4), assume that $f'(x) \leq 1-\varepsilon$. By definition of g' we have $f'(x) = g'(x)$, $h'(x) = 0$ in this case. Since $\operatorname{supp} h' \subset \{f' \geq 1\}$, the continuity of f' implies $\operatorname{dist}(x, \operatorname{supp} h') \geq \delta(\varepsilon)$. From (4), (8) and (11) we see that $w_x(x,t) = 0$ for $t \leq \delta/2$. The second assertion of (4.4) is proved by a similar argument. |||

<u>Nonuniqueness</u>. Clearly, properties (4.1) - (4.4) do not determine w uniquely. After all, for the construction of $\Pi(c_1,c_2,h)$, we could choose any interval $[c_1,c_2]$ that contains $\operatorname{supp} h'$. Also we may perturb the points $\bar{z}_{r1}$ which determine the discontinuity pattern of $w_x = \chi$. The discontinuities distinguish w from the smooth part v of the solution u and therefore we get a continuum of solutions for the problem (1).

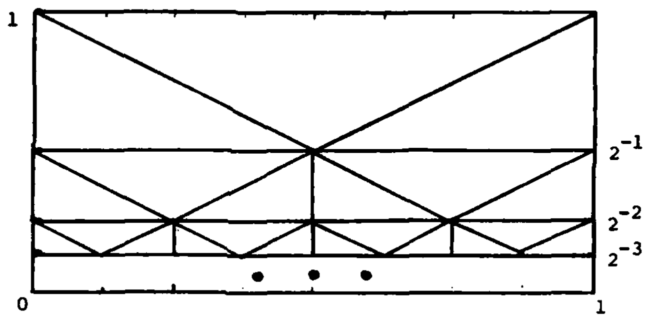
When reading our manuscript G. Strang pointed out to us a nice construction of a function w satisfying (4.1)-(4.4) which has a fixed discontinuity pattern, independent of the initial data h. His idea can be briefly described as follows.

Given h with $h(0) = 0$ and $0 \leq h'(x) \leq d$, $x \in [0,1]$, there exist continuous piecewise linear functions s_k with

$$s_k'\big|_{((j-1)2^{-k},\ j2^{-k})} \in \{0,d\}, \quad j = 1,\cdots,2^k$$

$$\|h - s_k\|_\infty \leq d \cdot 2^{-k} .$$

Consider now the following partition of $[0,1]^2$



and define w as the piecewise linear function with respect to this partition that agrees with s_k on the lines $(x,2^{-k})$, $x \in [0,1]$. It can be easily checked that w satisfies (4.1)-(4.4). Moreover the construction is not unique. Any scaling $t \mapsto \alpha t$ gives a different w with the same properties.

2. THE GENERAL CASE. As in the previous section we construct a solution u of the form $u = v + w$ where v is smooth and w is a function with oscillating derivative with respect to x. In view of the relation (3) we choose w of the form

$$w(v,x,\ ) = \int_0^x \chi(v,y,t)\,\left(A + B\, v_x(y,t)\right) dy\,, \tag{4'}$$

with $\chi : [0,1]\times[0,T] \to \{0,1\}$. To obtain a sufficiently regular equation for v it seems to be necessary to let χ, i.e. the discontinuity pattern of the solution u, depend on v. An appropriate choice of χ will yield $w \in C^\alpha$ and $w_t = B\,\chi\, v_t + \psi$ with $\psi \in L_\infty$. This choice of χ, and hence of w, leads to an equation for v of the form (cf. Proposition 2)

$$v_t + B\,\chi(v,\cdot)\, v_t + \psi(v,\cdot) = v_{xx}\,.$$

To solve it, we have to study the structure of the partition corresponding to χ and the dependence of w, χ, ψ on v.

We define the initial values for v and w by $g(0) = f(0)$, $h(0) = 0$ and

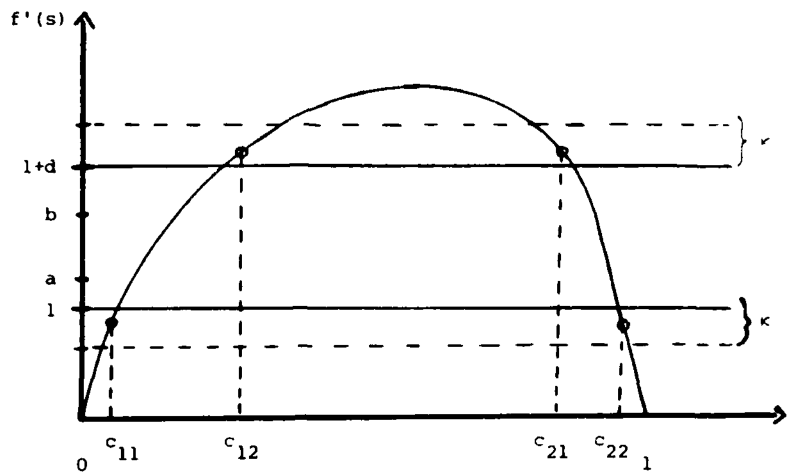
$$g'(x) = \begin{cases} f'(x) & , \ f'(x) \le 1 \\ 1 & , \ 1 \le f'(x) \le 1+d \\ (f'(x)-A)/(1+B) & , \ 1+d \le f'(x) \end{cases}$$

$$h'(x) = \begin{cases} 0 & , \ f'(x) \le 1 \\ f'(x)-1 & , \ 1 \le f'(x) \le 1+d\,. \\ (A+Bf'(x))/(1+B) & , \ 1+d \le f'(x) \end{cases} \tag{14}$$

Using $A + B = d$ (cf. p.4) one can easily check that g', h' are continuous, and from $f \in C^2[0,1]$ it follows that g'', $h'' \in L_\infty$.

In order not to complicate the proof of the Theorem by unessential technical details we assume that for the initial function f the set $\{1 \le f' \le 1+d\}$ consists of at most two intervals (cf. figure (14') below). In general we would have to carry out the constructions described below separately for a finite number of intervals of a suitably chosen partition of $[0,1]$.

(14')



<u>The partition</u> $\Pi(v)$. Throughout this and the following paragraph we fix a function v with $v, v_x \in C^\alpha$ that satisfies the initial condition $v(\cdot,0) = g$, g defined by (14,14').

Since $a > 1$, $A + B = d > 0$, $B > -1$ (cf. p.4) there is a constant κ with $\kappa < (a-1)$ such that

$$A + B\,[1-\kappa, 1+\kappa] \subseteq (\kappa, d+|B|\kappa) .$$

Let us assume that $\max g' > 1+\kappa$ because this is the slightly more complicated case for the construction of the partition $\Pi(v)$. We can find intervals $I_j = [c_{j1}, c_{j2}]$, $j=1,2$, and a constant $\kappa'(g')$ with $0 < \kappa' < \kappa$ and $2\kappa' < c_{21} - c_{12}$ so that

(15)
$$g'([c_{j1}-\kappa', c_{j2}+\kappa']) \subseteq (1-\kappa, 1+\kappa) , \quad j=1,2$$
$$g'([c_{12}, c_{21}]) \subseteq (1+\kappa', \infty) .$$

Moreover we may assume that c_{12}, c_{21} are of the form $c_{11} + \nu\, 2^{-N}(c_{22}-c_{11})$, where $N = N(f')$ is sufficiently large. Since $v_x \in C^\alpha$ <u>and</u> $v_x(\cdot,0) = g$ <u>there exists</u> $T(g', \|v_x\|_\alpha) > 0$ <u>such that for</u> $t \leq T$ (15) <u>remains valid with</u> g' <u>replaced by</u> $v_x(\cdot,t)$. This implies in particular

(16) $$A + B\,v_x(x,t) \in (\kappa, d+|B|\kappa) , \quad (x,t) \in I_j' \times [0,T], \quad j=1,2 ,$$

where $I_j' = [c_{j1} - \kappa', c_{j2} + \kappa']$, a fact we shall frequently use in the sequel.

LEMMA 1. <u>There exists</u> $T > 0$ <u>such that for</u> $t < T$ <u>the equation</u>

$$A\,x + B\,v(x,t) = A\,y + B\,g(y) \tag{17}$$

<u>defines two one to one maps</u>

$$\Lambda_j \;:\; I_j \times [0,T] \to \Omega_j \subseteq I_j' \times [0,T] \;:\; (y,t) \to (x,t)\;,\quad j=1,2\;,$$

<u>which are strictly monotone increasing in the first coordinate</u>.

<u>Proof</u>. Since $v \in C^\alpha$ and $v(\cdot,0) = g$ there exists $T(g,\|v\|_\alpha,\kappa') > 0$ such that for $t < T$

$$A\,I_j' + B\,v(I_j',t) \supseteq A\,I_j + B\,g(I_j)$$

which implies that (17) can be solved for $x \in I_j'$ if $y \in I_j$. To complete the proof we note that by (16) both sides of (17) are strictly increasing functions of their arguments x,y respectively. |||

For the definition of $\Pi(v)$ described below it is convenient to define a single map

$$\Lambda \;:\; [c_{11},c_{22}] \times [0,T] \to [0,1] \times [0,T]$$

which agrees with Λ_j on $I_j \times [0,T]$. To this end we set

$$ZL(t) = \Lambda_1(c_{12},t)\;,\quad ZR(t) = \Lambda_2(c_{21},t)\;,$$

i.e. ZL, ZR denote the right and left boundaries of Ω_1, Ω_2 respectively, and define

$$\Lambda(y,t) = \begin{cases} \Lambda_j(y,t)\;, & (y,t) \in I_j \times [0,T] \\ \dfrac{c_{21} - y}{c_{21} - c_{12}}\, ZL(t) + \dfrac{y - c_{12}}{c_{21} - c_{12}}\, ZR(t), & c_{12} < y < c_{21}\;. \end{cases} \tag{18}$$

As the individual maps Λ_j, Λ is one to one and strictly monotone increasing in the first coordinate. We cannot use (17) to define Λ since for $x \in (c_{12},c_{21})$ the function $Ax + Bv(x,t)$ needs not be monotone increasing and hence (17) may not be uniquely solvable for x.

The partition $\Pi(v)$ is a perturbation of the partition $\Pi(c_{11},c_{22},h_0)$ described in section 1 restricted to $[c_{11},c_{22}] \times [0,T]$ where the function h_0 is defined by

$$h_0(0) = 0\;,\quad h_0' := \min(h',d) = d\,h' / (A+Bg')$$

with h given by (14). Note that $\Pi(c_{11},c_{22},h_0)$ refers to the partition constructed from h_0 rather than from h. We keep the notation introduced in section 1, in particular we denote by z_r the lines determining the partition $\Pi(c_{11},c_{22},h_0)$. We define $\Pi(v)$ as the image of $\Pi(c_{11},c_{22},h_0)$ under the map Λ, i.e. we replace for $|r| > 2 - \log_2 T$ the lines z_r by the curves

$$
(19) \qquad \begin{aligned} z_r(v,\cdot) \; &: \; [2^{-|r|+1},2^{-|r|+2}] \to [0,1] \\ (z_r(v,t),t) \; &= \; \Lambda(z_r(t),t) \, . \end{aligned}
$$

Since the points c_{12},c_{21} had been chosen of the form $c_{11} + \nu\, 2^{-N}(c_{22} - c_{11})$ (we may assume $T < 2^{-N}$) the right and left boundaries of the rectangles $I_j\times[2^{-|r|+1},2^{-|r|+2}]$, $j=1,2$, agree with lines from the partition $\Pi(c_{11},c_{22},h_0)$. Therefore, for the definition of one particular curve $z_r(v,\cdot)$ Λ either coincides with Λ_j or is given by the second formula in (18). We denote by Ξ the set of all r for which the first possibility applies, i.e.

$$\Xi = \{r : z_r(t) \in I_1 \cup I_2 \, , \;\; t \in [2^{-|r|+1},2^{-|r|+2}]\} \, .$$

By Lemma 1 and the definition (18) of Λ, the partition $\Pi(v)$ has the same structure as $\Pi(c_{11},c_{22},h_0)$. By this we mean that

$$(7.1') \qquad \cdots \le z_r(v,\cdot) \le z_{r+1}(v,\cdot) \le \cdots$$

$$(7.2') \qquad \underline{z}_{r0}(v) = \overline{z}_{r00}(v)$$

$$(7.3') \qquad \underline{z}_{r1}(v) = \overline{z}_{r11}(v)$$

$$(7.4') \qquad \overline{z}_{r01}(v) = \overline{z}_{r10}(v) \;\; \in \;\; [\underline{z}_{r0}(v),\underline{z}_{r1}(v)] \, ,$$

where $\overline{z}_r(v)$, $\underline{z}_r(v)$ denote the upper and lower endpoints of the curves $z_r(v,\cdot)$. Note that we have equality in (7.1') iff we have equality in (7.1) (cf. also (11)). Since $h_0'([c_{12},c_{21}]) = d$ this implies in particular

$$z_{r1}(v,\cdot) = z_{(r+1)0}(v,\cdot), \quad r1 \notin \Xi \, .$$

The following Lemma shows that for $t \to 0$ the partition $\Pi(v)$ "converges" to $\Pi(c_{11},c_{22},h_0)$.

LEMMA 2. $|z_r(v,t) - z_r(t)| \leq C\, t^{\alpha}$.

Here and in the sequel C denotes various positive constants which may depend on f, α, $\|v\|_{\alpha}$, $\|v_x\|_{\alpha}$. Also, we shall always assume $|r| > 2 - \log_2 T$ so that the curves $z_r(v,\cdot)$ are well defined.

<u>Proof</u>. We may assume $r \in \Xi$. Writing (17) in the form

$$A\,(x-y) + B\,\big(v(x,t) - v(y,t)\big) + B\,\big(v(y,t) - g(y)\big) = 0$$

we obtain the estimate

$$|(x-y)\,\big(A + B\,v_x(\xi,t)\big)| \leq C\, t^{\alpha}$$

and the Lemma follows from (16). |||

LEMMA 3. $\|z_r(v,\cdot)\|_{\alpha} \leq C$, <u>uniformly in</u> r.

<u>Proof</u>. We may assume $r \in \Xi$ and to simplify the notation we set $x=z_r(v,t)$, $x'=z_r(v,t')$, $y=z_r(t)$, $y'=z_r(t')$. From (17) we see that

$$A\,(x-x') + B\,\big(v(x,t) - v(x',t)\big) + B\,\big(v(x',t) - v(x',t')\big) =$$
$$A\,(y-y') + B\,\big(g(y) - g(y')\big).$$

Writing $v(x,t) - v(x',t) = v_x(\xi,t)\,(x-x')$ and using (16) and the estimate $|g(y) - g(y')| \leq C\,|z_r'|\,|t-t'|$ finishes the proof. |||

<u>The functions</u> $\chi(v,\cdot)$ <u>and</u> $w(v,\cdot)$. Denote by R_r, R_r' the "rectangles"

$$R_r := \{(x,t) : z_{r0}(v,t) < x \leq z_{r1}(v,t)\ ,\ \ t \in [2^{-|r|}, 2^{-|r|+1}]\ \}\ ,$$
$$R_r' := \{(x,t) : z_{r1}(v,t) < x \leq z_{(r+1)0}(v,t)\ ,\ \ t \in [2^{-|r|}, 2^{-|r|+1}]\}\ .$$

Corresponding to the partition Π, constructed in the previous paragraph, we define the function χ by

$$\chi(v,x,t) = \begin{cases} 1, & (x,t) \in R_r \\ 0, & \text{otherwise.} \end{cases} \tag{8'}$$

From the remark following (7.4') we see that

(20) $\chi(v,x,t) = 1, \quad ZL(t) \le x \le ZR(t)$,

i.e. χ does not depend on the particular form of the curves $z_r(v,\cdot)$ for $r \notin \Xi$. We gave an explicit definition for these curves merely becuase then we do not have to treat each of the intervals I_1, I_2 separately.

Substituting (8') in the definition (4') for w we obtain

$$(12') \qquad w(v,x,t) = \begin{cases} \sum_{s<r} A\left(z_{s1}(t) - z_{s0}(t)\right) + B\left(g(z_{s1}(t)) - g(z_{s0}(t))\right) , \\ \qquad \text{if } (x,t) \in R_r' \\ \sum_{s<r} A\left(z_{s1}(t) - z_{s0}(t)\right) + B\left(g(z_{s1}(t)) - g(z_{s0}(t))\right) + \\ A\,x + B\,v(x,t) - A\,z_{r0}(t) - B\,g(z_{r0}(t)) , \\ \qquad \text{if } (x,t) \in R_r . \end{cases}$$

This follows from (17), (18). E.g. if $(x,t) \in R_r'$ and $0 \le x \le ZL(t)$ we have

$$w(v,x,t) = \Sigma \int_{z_{r0}}^{z_{r1}} \left(A + B\,v_x(y,t)\right) dy$$

$$= \Sigma\, A\left(z_{s1}(v,t) - z_{s0}(v,t)\right) + B\left(v(z_{s1}(v,t),t) - v(z_{s0}(v,t),t)\right)$$

$$= \Sigma\, A\left(z_{s1}(t) - z_{s0}(t)\right) + B\left(g(z_{s1}(t)) - g(z_{s0}(t))\right) .$$

Similarly we argue if $(x,t) \in R_r$ and $0 \le x \le ZL(t)$. If $ZL(t) \le z_{r1}(v,t) \le ZR(t)$ we have $z_{r1} = z_{(r+1)0}$ which implies $R_r' = \emptyset$. Therefore, if $x \in (ZL(t),ZR(t))$, $x \in R_r$,

$$\omega(v,x,t) = \omega(v,ZL(t),t) + \int_{ZL(t)}^{x} (A + Bv_x(y,t))\, dy$$

agrees with (12') because the terms $A\,z_{s1} + B\,g(z_{s1})$ and $A\,z_{(s+1)0} + B\,g(z_{(s+1)0})$ cancel. Finally if $x > ZR(t)$ we argue similarly as for $x \le ZL(t)$.

If $B = 0$ one can check that $h_0' = h$, $z_r(v,\cdot\,) = z_r$, $\Pi(v) = \Pi(c_{11},c_{22},h)$, $w_x = A\,\chi = d\,\chi$ which shows that our definition is consistent with the special case treated in section 1. Let us check that w satisfies the initial conditions (14).

LEMMA 4. $\lim_{t\to 0} \|w(v,\cdot,t) - h\|_\infty = 0$.

Proof. Since $|w_x| \le |A| + |B|\,|v_x| \le C$, it is sufficient to check the convergence for sufficiently many points. We shall show

$$w(v,z_{r00}(v,t),t) \longrightarrow h(z_{r00}(v,t)) , \quad |r| \to \infty ,$$

uniformly in $t \in [2^{-|r|-1}, 2^{-|r|}]$. In view of Lemma 2 we can replace $h(z_{r00}(v,t))$ by $h(z_{r00}(t)) = h(\underline{z}_{r0})$. From (12') we see that

$$w(v,z_{r00}(v,t),t) = \sum_{s<r0} A\left(z_{s1}(t) - z_{s0}(t)\right) + B\left(g(z_{s1}(t) - g(z_{s0}(t))\right) .$$

By definition of the partition $\Pi(c_{11},c_{22},h_0)$ (cf. (10)) we have

$$z_{s1}(t) - z_{s0}(t) = \bar{z}_{s1} - \bar{z}_{s0} = d^{-1} \int_s h_0' .$$

Using this, $A + B = d$ and twice the mean value theorem we obtain

$$\begin{aligned}\sum_{s<r0} \cdots &= \sum \left(A + B\, g'(\xi_s)\right) d^{-1} \int_s h_0' \\ &= \sum (\bar{z}_{(s+1)00} - \bar{z}_{s00}) \left(h_0'(\zeta_s)\, (A + B\, g'(\xi_s))/d\right) .\end{aligned}$$

This can be interpreted as a Riemann sum for $\int_{[c_{11},\underline{z}_{r0}]} h_0'\, (A+Bg')/d = \int_{[c_{11},\underline{z}_{r0}]} h'$ which proves the Lemma since $h'' \in L_\infty$. |||

From (12') we can formally compute w_t (cf. p.19 for a proof) which is given by

(13') $$w_t(v,x,t) = B\,\chi(v,x,t)\, v_t(x,t) + \psi(v,x,t)$$

where

(13'') $$\psi(v,x,t) = \begin{cases} \sum_{s<r} B\left(g'(z_{s1}(t)) - g'(z_{s0}(t))\right) z_{s0}' , & (x,t) \in R_r' \\ \sum_{s<r} B\left(g'(z_{s1}(t)) - g'(z_{s0}(t))\right) z_{s0}' - \left(A + B\, g'(z_{r0}(t))\right) z_{r0}' , & (x,t) \in R_r .\end{cases}$$

Since $g'' \in L_\infty$ we have $\psi \in L_\infty$ and, if $v_t \in L_2$, $w_t \in L_2$.

LEMMA 5. $\|w(v,\cdot)\|_{\alpha^2} \le C$.

Proof. Since $w_x \in L_\infty$ it is sufficient to prove the Hölder continuity with respect to t . In estimating $w(v,x,t) - w(v,x,t')$ let us first assume that $t,t' \in [2^{-j}, 2^{-j+1}]$.

We consider two cases

1. for some r, $|r|=j$, $(x,t),(x,t')\in R_r$. In this case it follows from (12') that

$$\begin{aligned}|w(v,x,t)-w(v,x,t')| &\le \Big|\sum_{s<r} B\big(g(z_{s1}(t))-g(z_{s1}(t'))\big)+\\ &\qquad B\big(g(z_{s0}(t'))-g(z_{s0}(t))\big)\Big|+\\ &\quad |B|\,|v(x,t)-v(x,t')|+A\,|z_{r0}(t)-z_{r0}(t')|+\\ &\quad |B|\,|g(z_{r0}(t))-g(z_{r0}(t'))|\\ &\le |t-t'|\,\Sigma|B|\,|g'(\xi_s)z'_{s1}-g'(\zeta_s)z'_{s0}|+\\ &\quad C\,|t-t'|^{\alpha}+C\,|t-t'|\\ &\le C\,|t-t'|^{\alpha}.\end{aligned}$$

The same estimate we obtain if $(x,t),(x,t')\in R'_r$.

2. $(x,t),(x,t')$ do not lie in a common "rectangle" R_r or R'_r. It follows from Lemma 2 that at most $C\,2^{j\alpha}$ of the curves $z_{r\nu}(v,\cdot)$ can intersect the segment $(x,[t,t'])$. Therefore we can find $t=t_0<t_1<\cdots<t_N=t'$, $N\le C\,2^{j\alpha}$ such that for each pair t_ν, $t_{\nu+1}$ either one of the previous cases applies. Hence we obtain the estimate

$$\begin{aligned}|w(v,x,t)-w(v,x,t')| &\le C\sum_{\nu=1}^{N}|t_\nu-t_{\nu-1}|^{\alpha}\\ &\le N^{1-\alpha}\,|t'-t|^{\alpha}\le C\,|t'-t|^{\alpha^2},\end{aligned}$$

where for the last inequality we used $N\le C\,|t'-t|^{-\alpha}$.

The general case follows now easily. For $t\in[2^{-j},2^{-j+1}]$, $t'\in[2^{-k},2^{-k+1}]$, $j<k$, we obtain, using the previous estimates,

$$\begin{aligned}|w(v,x,t)-w(v,x,t')| &\le C\Big(|t-2^{-j}|^{\alpha^2}+|2^{-k+1}-t'|^{\alpha^2}+\sum_{\nu=j+1}^{k-1}2^{-\nu\alpha^2}\Big)\\ &\le C\,|t-t'|^{\alpha^2}.\ |||\end{aligned}$$

__The dependence of__ Π,χ,w,ψ __on__ v. Throughout this paragraph, which is the final preparation for the proof of the Theorem, we shall restrict v to the set

(21) $$K := \{v : \|v\|_\alpha,\ \|v_x\|_\alpha,\ \|v_t\|_2\le C,\ v(\cdot,0)=g\}.$$

One can easily check that the constants in the previous Lemmas, in particular κ, κ' and T, can be chosen uniformly with respect to $v\in K$.

LEMMA 6. <u>The following maps are continuous.</u>

(22.1) $v \to z_r(v,\cdot) \; : \; (K, \|\ \|_\infty) \to C([2^{-|r|+1}, 2^{-|r|+2}])$

(22.2) $v \to \chi(v,\cdot) \; : \; (K, \|\ \|_\infty) \to L_2([0,1]\times[0,T])$

(22.3) $v \to \psi(v,\cdot) \; : \; (K, \|\ \|_\infty) \to L_2([0,1]\times[0,T])$

(22.4) $v \to w(v,\cdot) \; : \; (K, \|\ \|_\alpha, \|\partial_x\ \|_\alpha) \to C([0,1]\times[0,T])$.

<u>Proof of (22.1)</u>. We may assume $r \in \Xi$. For $v,v' \in K$ and the corresponding curves $x = z_r(v,\cdot)$, $x' = z_r(v',\cdot)$ we obtain from (17)

$$A\,(x-x') + B\,\big(v(x,t) - v(x',t)\big) + B\,\big(v(x',t) - v'(x',t)\big) = 0$$

which implies $|x-x'| \le C\,\|v-v'\|_\infty$ because of (16). |||

<u>Proof of (22.2)</u>. Since $\chi \in \{0,1\}$ we have

$$\|\chi(v,\cdot) - \chi(v',\cdot)\|_2^2 \le \epsilon + \|\chi(v,\cdot) - \chi(v',\cdot)\|^2_{2,[0,1]\times[\epsilon,T]} .$$

For any $\epsilon > 0$ only finitely many of the curves $z_r(v,\cdot)$ overlap the rectangle $[0,1]\times[\epsilon,T]$. This observation finishes the proof in view of the definition (8') of χ and (22.1). |||

We skip the <u>proof of (22.3)</u> which is a slight extension of this argument.

<u>Proof of (22.4)</u>. Since for $v \in K$, $w(v,\cdot,0) = h$, and $w \in C^\alpha$ it is sufficient to prove the continuity into $C([0,1]\times[\epsilon,T]$, any fixed $\epsilon > 0$. Skipping most of the subscripts we write $w(v,x,t) - w(v',x,t)$ in the form (cf. (4'))

$$\int_0^x \big(A\,(\chi(v)-\chi(v')) + B\,(\chi(v)-\chi(v'))\,v_x + B\,\chi(v')\,(v_x - v'_x) \big) .$$

From the definition of χ and (22.1) we see that

$\|\chi(v,\cdot,t) - \chi(v',\cdot,t)\|_2 \to 0$, as $\|v-v'\|_\infty \to 0$, uniformly for $t > \epsilon$, which finishes the proof. |||

With the aid of Lemma 6 we can now justify the formal computation of w_t.

<u>Proof of (13')</u>. Let us first assume that v_t is continuous. From (17) we see that this implies $z_r(v,\cdot) \in C^1([2^{-|r|+1}, 2^{-|r|+2}])$. Hence R_r, R_r' have a piecewise C^1 boundary and since w is continuous we may compute w_t separately on these "rectangles". In this case (13') is a direct consequence of definition (12') of w.

To finish the proof for $v_t \in L_2$ we choose a smooth approximating sequence $v^n \in K$, $v^n \to v$ and note that by Lemma 6 $w(v^n,\cdot) \to w(v,\cdot)$, $\chi(v^n,\cdot)\, v_t \to \chi(v,\cdot)\, v_t^n$, $\psi(v^n,\cdot) \to \psi(v,\cdot)$ in L_2. |||

We now give the proof of the Theorem which is based on the following Proposition.

PROPOSITION 2. <u>There exists</u> $T > 0$ <u>such that the problem</u>

$$(5')\qquad \begin{aligned} v_t + B\,\chi(v,\cdot)\, v_t + \psi(v,\cdot) &= v_{xx} \\ v_x(0,t) = v_x(1,t) &= 0 \\ v(\cdot,0) &= g \end{aligned}$$

<u>has a solution on the rectangle</u> $[0,1]\times[0,T]$ <u>satisfying</u>

$$\|v\|_\alpha\,,\quad \|v_x\|_\alpha\,,\quad \|v_t\|_2\,,\quad \|v_{xx}\|_2 \le C\,,$$

<u>where</u> C <u>and</u> α <u>depend on</u> g.

<u>Proof</u>. We solve (5') by iterating in the form

$$(*)\qquad v_t^n + B\,\chi(v^{n-1},\cdot)\, v_t^n + \psi(v^{n-1},\cdot) = v_{xx}^n$$

with boundary and initial conditions as in (5'). Since $B > -1$ we have $0 < \min(1,1/(1+B)) \le 1/(1+B\chi)$, $\|\psi\|_\infty \le C$ and we can apply Theorem A to get uniform bounds for $\|v^n\|_\alpha$, $\|v_x^n\|_\alpha$, $\|v_t^n\|_2$, $\|v_{xx}^n\|_2$, i.e. all iterates stay in a set of the form (21). We choose T small enough, so that for all n, $\chi(v^{n-1},\cdot)$, $\psi(v^{n-1},\cdot)$ are well defined, i.e. $T = T(K)$. By compactness we can select for $\alpha' < \alpha$ a subsequence, again denoted by v^n, for which (*) holds and $v^n \to v$, $v_x^n \to v_x$ in $C^{\alpha'}$, $v_t^n \to v_t$, $v_{xx}^n \to v_{xx}$ weakly in L_2. To pass to the limit in (*), i.e. to show that v is a solution of (5'), we note that by Lemma 6 all terms in the equation (*) converge weakly in L_2 to the

corresponding expressions in (5'). E.g. we have

$$\lim \int \chi(v^{n-1},\cdot)\, v_t^n\, \phi =$$

$$\lim \int \big(\chi(v,\cdot)\, \phi\big)\, v_t^n + \lim \int \big(\chi(v^{n-1},\cdot) - \chi(v,\cdot)\big)\, v_t^n\, \phi = \int \chi(v,\cdot)\, \phi\, v_t$$

where for the last step we used (22.2) and $\|v_t^n\, \phi\|_2 \le c$. |||

<u>Proof of the Theorem</u>. Let v be a solution of (5'). We claim that $u = v + w$ solves the problem (1).

From $\operatorname{supp} w \subseteq [c_{11},c_{22}]\times[0,T]$, (5') and Lemma 4 we see that u satisfies the correct boundary and initial conditions. By (13') and (5') we have $v_t + w_t = v_{xx}$. Therefore it remains to show

$$(*) \qquad v_x = \phi(v_x + w_x) = \phi(v_x + \chi\,(A + B\, v_x)) .$$

By the continuity of v we have for sufficiently small $T > 0$

$$v_x(\Omega_0 \cup \Omega_1) \subseteq (1 - \kappa,\ 1 + \kappa).$$

Moreover $\chi(([0,c_{11}] \cup [c_{22},1]) \times [0,T]) = 0$ and $\chi([c_{12},c_{21}] \times [0,T]) = 1$. Therefore (*) follows from (2) and (3) which imply

$$\phi(s) = s, \qquad s < a$$

$$\phi(s + A + Bs) = s, \qquad s > 2 - a.$$

The regularity assertions of the Theorem are consequences of Lemma 5 and Proposition 2. |||

<u>Nonuniqueness</u>. By definition (8') of χ , w_x is discontinuous across the curves z_r. These discontinuities distinguish w from v , the smooth part of the solution u . One way of obtaining a continuum of different choices for w is to perturb the partition $\Pi(c_{11},c_{22},h_0)$ and hence $\Pi(v)$ e.g. as follows. In the construction of these partitions we replace the intervals $[2^{-|r|},2^{-|r|+1}]$ by $[\lambda 2^{-|r|},\lambda 2^{-|r|+1}]$ with $\lambda = 1$.

ACKNOWLEDGEMENT. We would like to thank John Nohel and Carl de Boor for their advice and many fruitful discussions. Moreover, we like to thank Fred Sauer for helping us with the numerical computations.

APPENDIX.

THEOREM A. *For* $a, f \in L_\infty([0,1]\times R_+)$ *with* $0 < a_0 \le a$, *and* $g \in C^\beta([0,1])$, $\beta>1$, *with* $g'(0) = g'(1) = 0$, *the problem*

$$\text{(A)}\qquad \begin{aligned} u_t &= a\, u_{xx} + f\,, \quad (x,t) \in [0,1]\times R_+ \\ u_x(0,t) &= u_x(1,t) = 0 \\ u(\cdot,0) &= g \end{aligned}$$

has a unique solution satisfying

$$\|u\|_\alpha,\ \|u_x\|_\alpha,\ \|u_{xx}\|_2 \le C$$

where $\alpha > 0$ *and* C *depend on* a_0, $\|a\|_\infty$, $\|f\|_\infty$, $\|g\|_\beta$, β .

There seems to be no convenient reference for this result. Therefore we include an argument deducing it from results in [L].

Differentiating (A) with respect to x we obtain, with $v = u_x$

$$\text{(A')}\qquad \begin{aligned} v_t &= (a\, v_x)_x + f_x \\ v(0,t) &= v(1,t) = 0 \\ v(\cdot,t) &= g'\,. \end{aligned}$$

This is an equation of the form (1.1) in [L., p.134]. From Theorems 4.1, 4.2, 7.1, 10.1 in [L, pp.133-210] with $n=1$, $\nu=a_0$, $\mu=\|a\|_\infty$, $q=r=\infty$, $\mu_1=\max(\|a\|_\infty,\|f\|_\infty)$ it follows that (A') has a unique solution satisfying $\|v\|_\alpha$, $\|v_x\|_2 \le C$ where $\alpha > 0$ and C depend on the quantities above. (For our purposes there is no point in distinguishing between the Hölder continuity with respect to x and t .)

With v a solution of (A') we set

$$\text{(1)}\qquad u(x,t) = h(t) + \int_0^x v(y,t)\,dy\,.$$

Formally substituting this into (A) we find that

$$h'(t) + \int_0^x v_t(y,t)\,dy = (av_x)(x,t) + f(x,t)$$

and therefore

$$(2) \qquad h(t) = g(0) + \int_0^t \big((av_x)(x,\tau) + f(x,\tau)\big)\, d\tau - \int_0^x \big(v(y,t) - v(y,0)\big)\, dy \, .$$

For $\phi \in L_2$ we interpret $\int \phi(x,\tau)\, d\tau$ as $\lim \int \phi^n(x,\tau)\, d\tau$, where ϕ^n is a smooth approximating sequence, using the fact that the map

$$\phi \to \int_0^{\cdot} \phi(\cdot,\tau)\, d\tau \ : \ L_2 \to L_2$$

is continuous.

We now show that u defined by (1) with h given by (2) is a solution of (A). To justify the definition of h we have to show that the right hand side of (2) does not depend on x . To this end let ϕ be a test function with $\operatorname{supp} \phi \subseteq (0,1)\times(0,T)$ and define η by $\eta_t = \phi$, $\eta(\cdot,T) = 0$. Integrating by parts we obtain

$$\iint h\, \eta_{tx} = -\iint \big((a\, v_x)(x,t) + f(x,t)\big)\, \eta_x(x,t)\, dx\, dt + \iint \big(v(x,t) - v(x,0)\big)\, \eta_t(x,t)\, dx\, dt = 0 \, ,$$

i.e. $h_x = 0$. For the last equality we have used that v is a weak solution of (A') (cf. [L, p.136]) and $-\iint v(x,0)\, \eta_t(x,t)\, dx\, dt = \int v(x,0)\, \eta(x,0)\, dx$. From the definition of h one can now easily check that u is a unique solution of (A).

To see that h , and hence u , is Hölder continuous we write

$$|h(t') - h(t)| = \Big|\int_0^1 \big(h(t') - h(t)\big)\, dx\Big| \le$$

$$\Big|\int_t^{t'}\int_0^1 \big((a\, v_x)(x,\tau) + f(x,\tau)\big)\, dx\, d\tau\Big| + \int_0^1\int_0^x |v(y,t') - v(y,t)|\, dy\, dx$$

$$\le \big\|\, |a\, v_x| + |f|\, \big\|_2\, |t' - t|^{1/2} + C\, |t' - t|^{\alpha} \, . \ |||$$

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KH/db

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
2354

2. GOVT ACCESSION NO.
AD-A116 155

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
EXISTENCE OF INFINITELY MANY SOLUTIONS FOR A NONLINEAR PARABOLIC EQUATION

5. TYPE OF REPORT & PERIOD COVERED
Summary Report - no specific reporting period

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Klaus Höllig

8. CONTRACT OR GRANT NUMBER(s)
DAAG29-80-C-0041
MCS-7927062, Mod. 1

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Mathematics Research Center, University of Wisconsin
610 Walnut Street
Madison, Wisconsin 53706

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
1 - Applied Analysis

11. CONTROLLING OFFICE NAME AND ADDRESS

12. REPORT DATE
March 1982

13. NUMBER OF PAGES
24

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Parabolic equation, nonlinear, diffusion, nonmonotone, constitutive function, existence, nonuniqueness

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Let ϕ be a piecewise linear function which satisfies the condition $s\phi(s) \geq cs^2$, $c > 0$, $s \in \mathbb{R}$, and which is monotone decreasing on an interval $(a,b) \subset \mathbb{R}_+$. It is shown that for $f \in C^2[0,1]$ with $\max f' > a$ there exists a $T > 0$ such that the initial boundary value problem

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. Abstract (continued)

$$u_t = \phi(u_x)_x$$

$$u_x(0,t) = u_x(1,t) = 0$$

$$u(\cdot,0) = f$$

has infinitely many solutions u satisfying

$$\|u\|_\alpha,\ \|u_x\|_\infty,\ \|u_t\|_2 \leq C(f,\phi)$$

on $[0,1] \times [0,T]$.

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7-8